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Multiresolution weighted norm equivalences and applications

S. Beuchler[1,⋆], **R. Schneider**[2,⋆], **C. Schwab**[3,⋆⋆]

[1] Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Science, Altenberger Strasse 69, 4040 Linz, Austria;
e-mail: sven.beuchler@oeaw.ac.at
[2] Scientific Computing, Institute of Computer Science, Christian Albrecht University Kiel, Christian-Albrechts-Platz 4, 24098 Kiel, Germany; e-mail: rs@numerik.uni-kiel.de
[3] Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, 8092 Zürich, Switzerland; e-mail: schwab@math.ethz.ch

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Summary. We establish multiresolution norm equivalences in weighted spaces $L^2_w((0,1))$ with possibly singular weight functions $w(x) \geq 0$ in $(0,1)$. Our analysis exploits the locality of the biorthogonal wavelet basis and its dual basis functions. The discrete norms are sums of wavelet coefficients which are weighted with respect to the collocated weight function $w(x)$ within each scale. Since norm equivalences for Sobolev norms are by now well-known, our result can also be applied to weighted Sobolev norms. We apply our theory to the problem of preconditioning p-Version FEM and wavelet discretizations of degenerate elliptic and parabolic problems from finance.

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1 Introduction

A basic tool in wavelet analysis are norm equivalences in Sobolev and Besov spaces [8, 10, 24]. They play a crucial role in multilevel preconditioning (see

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Correspondence to: C. Schwab

e.g. [10,25]) and also in nonlinear approximation [14,7]. Accordingly, multilevel norm equivalences have been proved for many types of multiresolution bases in scales of Sobolev and Besov spaces. In these norm equivalences, the levels or scales of wavelet expansions are mimicking a Littlewood-Paley decomposition, exploiting more the frequency behaviour of the basis function. Norm equivalences in terms of wavelet expansions for Sobolev and Besov spaces have been proved by several authors. First proofs were based on techniques borrowed from Fourier analysis see e.g. [24] and references therein. We also refer to the articles [8,6] for surveys. Despite their practical importance weighted spaces where the weight is a function of the space variable, have not been considered to our knowledge. However, the local support of the wavelet basis is especially suited to analyze the impact of the weight function $w(x)$ on the norm equivalence. To prove multilevel norm equivalences in scales of weighted Sobolev spaces with regular or singular weight function $w(x)$ is the purpose of the present paper.

The proof of such norm equivalences can not be based on explicit Fourier techniques due to the lack of translation invariance induced by the weight functions. Alternative proofs of norm equivalences are based exclusively on approximation theory, namely the inverse and the approximation property, respectively, and its relation with Besov norms [25,10]. Our proof of weighted norm equivalences is based on a strengthened Cauchy Schwarz inequality, a technique borrowed from domain decomposition and applied to multilevel preconditioning by [3]. With these techniques we prove an upper estimate [29] while the lower estimate can be easily deduced from the upper estimate for the dual wavelet basis in a biorthogonal setting like in [29]. For this reason we consider in our proofs the primal and dual wavelet systems simultaneously. We note that the singularity of the weight must be compensated in certain cases by homogeneous Dirichlet boundary conditions for the dual wavelet basis.

We consider several applications of our theory, in particular wavelet preconditioning of the element stiffness matrices for the p- or spectral FEM and the preconditioning of stiffness matrices from stochastic volatility models in finance. Here, the natural weights are the Jacobi weights which are singular at the boundary. Further applications of the present tools include weighted L^p-spaces or weights with singularities in the interior which are not considered explicitly here.

Let us briefly elaborate on the significance of preconditioning the elemental stiffness matrices in p-FEM, or when combined with mesh-refinement, in the hp-FEM. The hp-FEM applied to elliptic and parabolic problems allows for exponential convergence rates, in terms of the number of degrees of freedom, since the solutions are piecewise analytic [30,28]. Due to the cost in generating the element stiffness and mass matrices in hp-FEM and the numerical solution of the linear systems, in practical applications, in particular in three dimensions, the gain in using high polynomial degrees is in part offset by the computational expense in matrix generation and solution. Matrix generation in high order FEM can be accelerated to near optimal complexity by sum factorization and spectral quadrature techniques, see e.g. [31,23]. This leaves the numerical solution of the linear systems as computational bottleneck. Once the internal degrees of freedom on each element are condensed, effective iterative methods are available for the solution of the global linear systems (based e.g. on domain decomposition). In dimension three and for degree $p \geq 4$, however, the condensation process becomes extremely expensive, even if executed in parallel due to mutual independence of the internal degrees of freedom. Alternatively to condensation by direct solution (elimination), condensation by iterative methods could be considered. For efficiency, a preconditioner is required, since at high polynomial degree p, the element matrices can be rather ill-conditioned. p-element pre-conditioners were constructed early by spectrally equivalent low order finite - difference or finite element discretizations on graded tensor product meshes on Lobatto points (see [20], [15],).

Our norm equivalences suggest a different approach: we build a preconditioner based on wavelet discretizations on uniform meshes, but with the singular weights taken into account in each scale. We deduce from our weighted norm equivalences by judicious choice of the weights a new, spectrally equivalent wavelet preconditioner for the p-version FEM. In addition, the regular refinements of the sequence of grids and the dyadic structure of the wavelet basis allow for fast realization of this preconditioner. We close the paper by generalizing the weighted norm equivalences from L^2 to Sobolev spaces of nonzero order and present optimal wavelet preconditioners for multilevel FEM applied to a class of degenerate elliptic equations of second order.

The outline of the paper is as follows: In section 2, we present some background material about multiresolutions and wavelet bases. Section 3 contains the main technical tool of the paper, the discrete norm equivalences in weighted L^2 and higher order norms. Section 4 presents the construction of the preconditioner for the p-FEM, and Section 5 concludes with applications to anisotropic and degenerate elliptic problems.

2 Wavelets and multiresolution analysis

Multiresolution analysis is by now a well established tool in signal processing. Among the many excellent accounts, we refer the reader to the survey paper [9] and the references therein. Here we collect only some facts which are useful for our purpose. We need wavelets on the unit interval [0, 1]. There are different approaches to define wavelets on a finite interval. Our present method is based on the construction of orthogonal compactly supported wavelets on [0, 1] given in [7] and biorthogonal wavelets [11]. A multiresolution analysis on the interval [0, 1] consists of a nested family of finite dimensional subspaces

$$\mathbb{V}_0 \subset \mathbb{V}_1 \subset \ldots \subset \mathbb{V}_j \subset \mathbb{V}_{j+1} \subset \ldots \subset L^2((0,1)), \tag{2.1}$$

such that $\dim \mathbb{V}_l \sim 2^l$ and $\overline{\bigcup_{l\in\mathbb{N}_0} \mathbb{V}_l} = L^2((0,1))$ with $\mathbb{N}_0 = \{0, 1, \ldots\}$.

Each space $\mathbb{V}_l$ is defined by a single scale basis $\Phi_l = \{\varphi_k^l\}$, i.e., $\mathbb{V}_l = \operatorname{span}\{\varphi_k^l : k \in \Delta_l\}$, where Δ_l denotes a suitable index set with cardinality $\#(\Delta_l) \sim 2^l$. An important requirement is that these bases are uniformly stable, i.e., for any vector $c = \{c_k,\ k \in \Delta_l\}$

$$\|c\|_{l_2(\Delta_l)} \sim \left\| \sum_{k\in\Delta_l} c_k \varphi_k^l \right\|_0$$

holds uniformly in j. Furthermore, the single scale bases satisfy a locality condition

$$\operatorname{diam}\operatorname{supp}(\varphi_k^l) \sim 2^{-l}.$$

Instead of using only a single scale l one is interested in the supplement of information between an approximation of a function in the spaces $\mathbb{V}_l$ and $\mathbb{V}_{l+1}$. Since $\mathbb{V}_l \subset \mathbb{V}_{l+1}$ there are several ways to decompose $\mathbb{V}_{l+1} = \mathbb{V}_l \oplus \mathbb{W}_l$, with some complementary space $\mathbb{W}_l$, $\mathbb{W}_l \cap \mathbb{V}_l = \{0\}$, not necessarily orthogonal to $\mathbb{V}_l$. The complementary spaces W_e of $\mathbb{V}_l$ in $\mathbb{V}_{l+1}$ are spanned by the multi scale bases $\Psi_l = \{\psi_k^l : k \in \nabla_l = \Delta_{l+1}/\Delta_l\}$. It is supposed that the collections $\Phi_l \cup \Psi_l$ are also uniformly stable bases of $\mathbb{V}_{l+1}$. If $\Psi = \bigcup_{l=-1}^{\infty} \Psi_l$, where $\Psi_{-1} = \Phi_0$, is a Riesz–basis of $L^2((0,1))$ we will call it a wavelet basis. We consider basis functions ψ_l^j to be local with respect to the corresponding scale l, i.e., $\operatorname{diam}\operatorname{supp}\psi_k^l \le C_\psi 2^{-l}$ and we will normalize them by $\|\psi_k^l\|_{L_2((0,1))} \sim 1$. An important property of these functions are the vanishing moment property

$$\int_0^1 x^\alpha \psi_k^l(x)\,\mathrm{d}x = 0, \quad \text{for } \alpha = 0, 1, \ldots, \widetilde{d}\,. \tag{2.2}$$

In the dual space $\widetilde{\mathbb{W}}^l$ we have

$$\int_0^1 x^\alpha \widetilde{\psi}_k^l(x)\,\mathrm{d}x = 0, \quad \text{for } \alpha = 0, 1, \ldots, d\,. \tag{2.3}$$

We suppose that there exists also a biorthogonal, or dual, Riesz–basis

$$\widetilde{\Psi} = \{\widetilde{\psi}_k^l : k \in \nabla_l,\ l = -1, 0, 1, \ldots\} \subset L^2((0,1))$$

such that $\langle \widetilde{\psi}_k^l, \psi_j^i \rangle = \delta_{k,j}\delta_{i,l}$ and every $v \in L^2((0,1))$ has a representation

$$v = \sum_{l=-1}^{\infty} \sum_{k \in \nabla_l} \langle v, \psi_k^l \rangle \widetilde{\psi}_k^l = \sum_{l=-1}^{\infty} \sum_{k \in \nabla_l} \langle v, \widetilde{\psi}_k^l \rangle \psi_k^l \tag{2.4}$$

and that the norm equivalence

$$\|v\|_0^2 \sim \sum_{l=-1}^{\infty} \sum_{k \in \nabla_l} |\langle v, \psi_k^l \rangle|^2 \sim \sum_{l=-1}^{\infty} \sum_{k \in \nabla_l} |\langle v, \widetilde{\psi}_k^l \rangle|^2$$

holds. We refer to [9] for further details.

If one is going to use the spaces $\mathbb{V}_l$ and $\widetilde{\mathbb{V}}_l = \operatorname{span}\{\widetilde{\psi}_k^i : k \in \nabla_i, i = -1, 0, 1, \ldots, l-1\}$ as multiresolution spaces then additional properties are required for our purpose. We suppose that the following Jackson and Bernstein type estimates, respectively approximation and inverse property, hold for $t \le \tau \le d$, $t \le s < \gamma_0$ and uniformly in l

$$\inf_{v \in \mathbb{V}_l} \|u - v\|_t \le c 2^{-l(\tau - t)} \|u\|_\tau, \quad u \in H^\tau, \tag{2.5}$$

and

$$\|v\|_s \le c 2^{l(s-t)} \|v\|_t, \quad v \in \mathbb{V}_l, \tag{2.6}$$

where $\gamma_0, d > 0$ are fixed constants given by

$$\begin{aligned} \gamma_0 &= \sup\{s \in \mathbb{R} : \mathbb{V}_l \subset H^s((0,1))\}, \\ d &= \sup\{s \in \mathbb{R} : \ \text{ex.} b_0 > 0, \ \ \forall l \ge 0, u \in C^\infty : \\ &\qquad \inf_{v \in \mathbb{V}_l} \|u - v\|_0 \le b_0 2^{-ls} \|u\|_s\}. \end{aligned}$$

Usually, d is the maximal degree of polynomials which are locally contained in $\mathbb{V}_l$ and is referred to as order of exactness of the multiresolution analysis $\{\mathbb{V}_l\}$. The parameter γ_0 denotes the regularity or smoothness of the functions in the spaces $\mathbb{V}_l$. We will assume that $\gamma_0 \le d$, which is the case in all known examples of wavelet functions. Analogous estimates are supposed to be valid for the dual multiresolution analysis $\{\widetilde{\mathbb{V}}_l\}$ with constants $\widetilde{\gamma}_0, \widetilde{d}$.

Beside their importance in the approximation theory, the inequalities (2.5), (2.6) play a fundamental rule to establish norm equivalences, [8]. They provide a convenient device for switching between the norms $\|\cdot\|_t$ and corresponding sums of weighted wavelet coefficients from the representation (2.4). In fact the following norm estimates are a consequence of the approximation and the inverse inequality

$$\|v\|_t^2 \le c \sum_{l=-1}^{\infty} 2^{2lt} \sum_{k \in \nabla_l} |v_{l,k}|^2, \tag{2.7}$$

where $v = \sum_{l=-1}^{\infty} \sum_{k\in\nabla_l} v_{l,k}\psi_k^l$ and $v_{l,k} = \langle v, \widetilde{\psi}_k^l\rangle$ and $t < \gamma_0$,

$$\|v\|_t^2 \le c \sum_{l=-1}^{\infty} 2^{2lt} \sum_{k\in\nabla_l} |\widetilde{v}_{l,k}|^2 \tag{2.8}$$

where $v = \sum_{l=-1}^{\infty} \sum_{k\in\nabla_l} v_{l,k}\widetilde{\psi}_k^l$ and $\widetilde{v}_{l,k} = \langle v, \psi_k^l\rangle$ and $t < \widetilde{\gamma}_0$. We note that by a simple duality argument there follows the well known norm equivalence

$$\|v\|_t^2 \sim \sum_{l=-1}^{\infty} 2^{2lt} \sum_{k\in\nabla_l} |w_{l,k}|^2 , \tag{2.9}$$

for $t \in (-\widetilde{\gamma}_0, \gamma_0)$ if $w_{l,k} = \langle v, \widetilde{\psi}_k^l\rangle$. In the case $w_{l,k} = \langle v, \psi_k^l\rangle$ the above norm equivalence holds for $t \in (-\gamma_0, \widetilde{\gamma}_0)$, see, e.g., [8] and [29] for the details.

As a technical assumption for proving such norm equivalence in the case of weighted spaces we need that the wavelets and also the dual wavelets belong to $W^{1,\infty}((0, 1))$. This is satisfied for various families of spline wavelets constructed by stable completions, for example. In order that the wavelets together with their duals belong to the weighted function space, we also need a decay condition at the end points. Presently, we consider subsets $\mathbb{V}_l^0 \subset H_0^1((0, 1))$ of which satisfy homogeneous Dirichlet boundary conditions. For the spaces under consideration the index sets Δ_l can be characterized by the knots $\Delta_l = \{k2^{-l} : k = 0, \dots , 2^l\}$ or simply by $\{k = 0, \dots , 2^l\}$ and $\nabla_l = \{(k+1/2)2^{-l} : k = 0, \dots , 2^l-1\}$ or simply by $\{k = 1, \dots , 2^l\}$. It was shown in [12] that there are bases in $\mathbb{V}_l$ and $\widetilde{\mathbb{V}}_l$ such that $\phi_k^l(0) = \delta_{0,k}$ and $\widetilde{\phi}_k^l(0) = \delta_{0,k}$ and vice versa at the other end point. As indicated in [12] one removes the basis functions ϕ_0^l, $\widetilde{\phi}_0^l$, $\phi_{2^l}^l$ and $\widetilde{\phi}_{2^l}^l$ to define the subspaces $\mathbb{V}_l^0 := \text{span}\,\{\phi_k^l : k=1, \dots , 2^l - 1\}$ and $\widetilde{\mathbb{V}}_l^0 := \text{span}\{\widetilde{\phi}_k^l : k=1, \dots , 2^l - 1\}$. Obviously, all basis functions are zero at the end points. This choice induces other wavelet spaces $\mathbb{W}_l^0$ and wavelet bases $\{\psi_k^l\}$ (see [12] for further details). The only difference is that at the end points there are two basis functions ψ_k^l with $k = 1$ and $k = 2^{l-1}$ for which $\int_0^1 \psi_k^l(x)\,\mathrm{d}x \neq 0$.

For notational convenience we introduce

$$\nabla_l^I = \{k \in \mathbb{N}, 1 \le k \le 2^l - 1, 0 \notin \operatorname{supp} \psi_k^l\}$$

as the index set corresponding to all wavelets ψ_k^l which have a support with a positive distance to 0 and

$$\nabla_l^L = \{k \in \mathbb{N}, \beta - 1 \le k \le 2^l - 1, 0 \in \operatorname{supp} \psi_k^l\},$$

as the index set corresponding to all wavelets ψ_k^l having a support containing 0, and the parameter $\beta \in \mathbb{N}$ is specified later. Moreover, let $\widetilde{\nabla}_l^L = \{k \in \mathbb{N}, \beta - 1 \le k \le 2^l - 1, 0 \in \operatorname{supp} \widetilde{\psi}_k^l\}$.

3 Condition number of the mass matrix

Using (2.9), we have in particular

$$\|v\|_0 \equiv \sum_{l=1}^{\infty} \sum_{k\in\nabla_l} \mid w_{l,k} \mid^2 .$$

In this section, we prove an estimate for the condition number of the mass-matrix M of a weighted L^2_w norm given by

$$\begin{aligned} M &= \left(\frac{\int_0^1 w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x}{w(2^{-l}k)w(2^{-l'}k')} \right)_{(k,l);(k',l')} \\ &:= \left(\left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right)_{(k,l);(k',l')} \end{aligned} \tag{3.1}$$

in a multiresolution basis $\{\psi_k^l\}_{(k,l)}$ with the following properties

- The wavelets ψ_k^l and their duals are normed such that $\| \psi_k^l \|_{L^1} = C_\psi 2^{-\frac{l}{2}}$ holds.
- The wavelets have a vanishing moment condition, e.g. $\int_0^1 \psi_k^l(x)\,\mathrm{d}x = 0$.

We split the main result into several lemmas. Throughout, we make the following two assumptions.

Assumption 3.1 *The nonnegative weight function $w(x)$ is assumed to belong to $W^{1,\infty}((\delta,1))$ for every $\delta > 0$ and to satisfy*

$$C_w^{-1} \le \frac{w(x)}{x^\alpha} \le C_w, \quad C_w^{-1} \le \frac{w'(x)}{x^{\alpha-1}} \le C_w,$$

for some $C_w > 0$ and some $\alpha \in \mathbb{R}$.

Here and in the following, C_w denotes a generic positive constant depending only on the weight function $w(x)$ which can take different values in different places. The parameter α will be specified in the next assumption.

For the wavelets ψ_k^l near $x = 0$, we assume the following kind of multiresolution spaces.

Assumption 3.2 $\psi_k^l \in \mathbb{W}^0 \subset W^{1,\infty}((0,1))$ *and* $\widetilde{\psi}_k^l \in \widetilde{\mathbb{W}}^0 \subset W^{1,\infty}((0,1))$ *satisfy*

$$\begin{aligned} &|\psi_k^l(x)| \le C_\psi 2^{l/2}(2^l x)^\beta, \\ &|(\psi_k^l)'(x)| \le C_\psi 2^{3l/2}(2^l x)^{\beta-1}, x \in [0,2^{-l}],\ \beta \in \mathbb{N}_0,\ k \in \nabla_l^L \\ &|\widetilde{\psi}_k^l(x)| \le C_\psi 2^{l/2}(2^l x)^{\widetilde{\beta}}, \\ &|(\widetilde{\psi}_k^l)'(x)| \le C_\psi 2^{3l/2}(2^l x)^{\widetilde{\beta}-1}, x \in [0,2^{-l}],\ \widetilde{\beta} \in \mathbb{N}_0,\ k \in \widetilde{\nabla}_l^L. \end{aligned} \tag{3.2}$$

We assume that $\alpha + \beta > -\frac{1}{2}$ and $-\alpha + \widetilde{\beta} > -\frac{1}{2}$.

Remark 3.1 The estimate (3.2) is only required for boundary wavelets, that is $k = 1, ..., N$. We write $k \approx 1$ in this situation. The boundary wavelets ψ_k^l with $k \approx 1$ satisfy homogeneous Dirichlet boundary conditions up to order β. Constructions of such boundary wavelets can be found for example in [12,5].

We note further that these functions generally do not satisfy vanishing moment conditions.

We assume throughout that our wavelets have compact support, in particular that

$$\mathrm{supp}(\psi_1^0) \subseteq [0, 2N-1].$$

Furthermore, the parameter C_ψ is a constant which is independent of the level numbers l and l', and, k and k'.

We state now two technical lemmas required in order to estimate the weight function. The results can be proved by simple estimates.

Lemma 3.1 *Let $\xi, 2^{-l'}k' \in [2^{-l}(k-N), 2^{-l}(k+N)]$ and $N \in \mathbb{N}$ with $0 < N < k$. Then, the weight function w satisfies*

$$\frac{w^2(\xi)}{w(2^{-l}k)w(2^{-l'}k')} < C_w$$

uniformly with respect to l and k.

Lemma 3.2 *Let k', ξ and w satisfy the assumptions of Lemma 3.1 and let $l < l'$. Then there holds*

$$\left| 2^{-l} \frac{[w^2]'(\xi)}{w(2^{-l}k)w(2^{-l'}k')} \right| < C_w.$$

We are now in position to prove the strengthened Cauchy-Schwarz inequalities. We consider first the situation when $0 \notin \mathrm{supp}\, \psi_k^l$. We assume that $l' \geq l$.

Proposition 3.1 *If $l = l'$ and $0 \notin \mathrm{supp}\, \psi_k^l \cup \mathrm{supp}\, \psi_{k'}^{l'}$, then*

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq C_\psi C_w. \tag{3.3}$$

Proof. The proof is standard. □

We prove now an estimate for $|(\psi_k^l, \psi_{k'}^{l'})_w|$, $l' > l$, in the case that ψ_k^l has a support not containing 0.

Lemma 3.3 *Let $l' > l$, $0 \notin \mathrm{supp}\, \psi_k^l$ and $\psi_k^l \in W^{1,\infty}(\mathrm{supp}\, \psi_{k'}^{l'})$. If $\mathrm{supp}\, \psi_k^l \cap \mathrm{supp}\, \psi_{k'}^{l'} \neq \emptyset$ then*

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq C_\psi C_w 2^{-\frac{3}{2}(l'-l)}.$$

Proof. See Appendix. □

Remark 3.2 If $l' > l$ and $0 \in \operatorname{supp} \psi_k^l$, but $k' > 2^{l'-l}$, the result

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \le C_\psi C_w 2^{-\frac{3}{2}(l'-l)}$$

follows by the same arguments.

Next, we consider the case that $0 \in \operatorname{supp} \psi_k^l$, but $0 \notin \operatorname{supp} \psi_{k'}^{l'}$, $l' > l$ and $k' < 2^{l'-l}$.

Lemma 3.4 *Let $l' > l$, $0 \in \operatorname{supp} \psi_k^l$ and $0 \notin \operatorname{supp} \psi_{k'}^{l'}$. If $0 < k' < 2^{l'-l}$ then*

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \le C_w C_\psi 2^{-\frac{1}{2}(l'-l)(1+2\alpha+2\beta)} k'^{\alpha+\beta-1}.$$

Proof. See Appendix. □

From now on, we do not distinguish C_w, C_ψ and absorb all constants into a generic c which is independent of l, l', k, k'.

Summing up the estimate in Lemma 3.4 over all $k' = 1, \dots, 2^{l'-l}$, the next lemma follows immediately.

Lemma 3.5 *Let $l' > l$ and $0 \in \operatorname{supp} \psi_k^l$, $0 \notin \operatorname{supp} \psi_{k'}^{l'}$. Then*

$$\sum_{k'=1}^{2^{l'-l}} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \le c \begin{cases} 2^{-\frac{1}{2}|l'-l|} & \text{if } \alpha+\beta \neq 0 \\ 2^{-\frac{1}{2}|l'-l|}|l'-l| & \text{if } \alpha+\beta = 0 \end{cases}.$$

In the extreme case $0 \in \operatorname{supp} \psi_k^l \cap \operatorname{supp} \psi_{k'}^{l'}$, we note that $k' \approx 1$. Then, we obtain a similar estimate as in Lemma 3.4.

Lemma 3.6 *Let $l' > l$ and $0 \in \operatorname{supp} \psi_k^l \cap \operatorname{supp} \psi_{k'}^{l'}$. Then, there holds*

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \le c 2^{-\frac{1}{2}|l'-l|(1+2\alpha+2\beta)}.$$

Proof. See Appendix. □

Next, we prove the boundedness of $M = ((\psi_k^l, \psi_{k'}^{l'}))_{k,l;k',l'}$ in l^2 using the well known Schur lemma. For this purpose, the next proposition determines the number of nonzero entries for the matrix M.

Proposition 3.2 *For fixed integer $l' > l$ each row of the block matrix $M_{l,l'} = ((\psi_k^l, \psi_{k'}^{l'}))_{l,l'}$ contains at most $\mathcal{O}(2^{l-l'})$ nonzero entries while the columns contain at most $\mathcal{O}(1)$ nonzero matrix entries.*

Proof. The assertion follows directly from the properties of hierarchical basis functions, cf. [29]. □

For wavelets ψ_k^l, $k \in \nabla_l^I$, we prove now the boundedness of the corresponding block of the mass matrix. We start with the case $0 \notin \operatorname{supp} \psi_k^l \cap \operatorname{supp} \psi_{k'}^{l'}$.

Theorem 3.1 *The estimate*

$$\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^I} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l}{2}} \le c 2^{-\frac{l'}{2}} \quad k' \in \mathbb{N}$$

is valid.

Proof. Let $k \in \nabla_l^I$ and $k' \in \nabla_{l'}^I$. Then it follows by Lemma 3.3 and Proposition 3.2

$$\begin{aligned}
&\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^I} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l}{2}} \\
&\quad \le c \sum_{l=1}^{\infty} \sum_{k \in \nabla_l^I} 2^{-\frac{l}{2}} 2^{-\frac{3}{2}|l-l'|} \delta_{\operatorname{supp} \psi_k^l, \operatorname{supp} \psi_{k'}^{l'}} \\
&\quad \le c \left(\sum_{l=1}^{l'} 2^{-\frac{3}{2}(l'-l)} 2^{-\frac{l}{2}} + \sum_{l=l'+1}^{\infty} 2^{-\frac{3}{2}(l-l')} 2^{-\frac{l}{2}} 2^{l-l'} \right) = c 2^{-\frac{l'}{2}},
\end{aligned}$$

where $\delta_{E,E'} = 0$ if two intervals E and E' satisfy $meas(E \cap E') = 0$ and $\delta_{E,E'} = 1$ otherwise. Consider now the case $k' \in \nabla_{l'}^L$. For $l < l'$ there holds

$$\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| = 0 \quad k \in \nabla_l^I, k' \in \nabla_{l'}^L \tag{3.4}$$

and we estimate

$$\begin{aligned}
\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^I} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l}{2}} &= \sum_{l=l'}^{\infty} \left(\sum_{k=1}^{2^{l-l'}} + \sum_{k > 2^{l-l'}} \right) \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l}{2}} \\
&=: A_1 + A_2.
\end{aligned} \tag{3.5}$$

We apply now Lemma 3.5 to estimate the first sum A_1 of (3.5) by

$$A_1 = \sum_{l=l'}^{\infty} 2^{-\frac{l}{2}} 2^{\frac{l'-l}{2}} (l - l') = 2^{-\frac{l'}{2}} \sum_{l=l'}^{\infty} 2^{l'-l} (l - l') = 2^{-\frac{l'}{2}} \sum_{l=0}^{\infty} 2^{-l} l = c 2^{-\frac{l'}{2}}$$

for $\alpha + \beta = 0$ and

$$A_1 = \sum_{l=l'}^{\infty} 2^{-\frac{l}{2}} 2^{\frac{l'-l}{2}(2\alpha+2\beta+1)} = 2^{-\frac{l'}{2}} \sum_{l=l'}^{\infty} 2^{(l'-l)(\alpha+\beta+1)} = c 2^{-\frac{l'}{2}}$$

for $\alpha + \beta \ne 0$ and $\alpha + \beta > -1$. The second term A_2 of (3.5) can be handled as in the case of $k' \in \nabla_{l'}^I$, cf. Remark 3.2. $\square$

Remark 3.3 The same proof allows also to obtain the estimate

$$\forall k' \in \mathbb{N}: \quad \sum_{l=1}^{\infty} \sum_{k \in \nabla_l^I} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c$$

Next, we consider the case $k \in \nabla_l^L$ and $k' \in \nabla_{l'}^I$.

Lemma 3.7 *There holds*

$$\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^L} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \quad k' \in \nabla_{l'}^I.$$

Proof. See Appendix. □

Remark 3.4 For the sums

$$2^{\frac{l}{2}} \sum_{l'=1}^{\infty} \sum_{k' \in \nabla_{l'}^L} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l'}{2}}$$

the estimates can be obtained in the same way. We obtain only a different bound in the case $\alpha + \beta = 0$ since we have a summation over 1s rather than a convergent series in (A.8). There holds

$$\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^L} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| 2^{-\frac{l}{2}} \leq c \begin{cases} 2^{-\frac{l'}{2}} & \text{if } \alpha + \beta \neq 0 \\ l' 2^{-\frac{l'}{2}} & \text{if } \alpha + \beta = 0 \end{cases} \quad k' \in \nabla_{l'}^I.$$

The last case to be considered is $k \in \nabla_l^L$ and $k' \in \nabla_{l'}^L$.

Lemma 3.8 *There holds*

$$\sum_{l=1}^{\infty} \sum_{k \in \nabla_l^L} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \quad k' \in \nabla_{l'}^L.$$

Proof. We note that on each level l not more than $\mathcal{O}(1)$ wavelets ψ_k^l satisfy $0 \in \operatorname{supp} \psi_k^l$. Therefore the summation over $k \in \nabla_l^L$ is done over not more than $\mathcal{O}(1)$ scalar products $(\psi_k^l, \psi_{k'}^{l'})_w$. By Lemma 3.6 we have the following estimate

$$\sum_{l=0}^{\infty} \sum_{k \in \nabla_l^L} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \sum_{l=0}^{\infty} 2^{-\frac{1}{2}|l'-l|(1+2\alpha+2\beta)} \leq c$$

iff $1 + 2\alpha + 2\beta > 0$. □

Now, we are able to formulate the main results of this section.

Theorem 3.2 *The infinite matrix* $M = ((\psi_k^l, \psi_{k'}^{l'})_w)_{(k,l);(k',l')}$ *is bounded in* l_2.

Proof. We decompose the matrix M into $M = M_1 + M_2$ where the coefficients in M_2 are $(\psi_k^l, \psi_{k'}^{l'})_w$ iff $0 \in \operatorname{supp} \psi_k^l \cap \operatorname{supp} \psi_{k'}^{l'}$ and M_1 does not contain the interaction of wavelets which are both located at the point zero. By applying Theorem 3.1, Lemma 3.7 and the Schur Lemma to M_1 we have $\| M_1 \|_2 \leq c$. From Lemma 3.8 we have $\| M_2 \|_1 \leq c$ and $\| M_2 \|_\infty \leq c$ which shows $\| M_2 \|_2 \leq c$. Hence, the assertion is proven. □

We show now the equivalence of the L_w^2 norm of a function

$$u = \sum_{l=l_0}^{\infty} \sum_k u_k^l \psi_k^l \in L_w^2\,((0,1))$$

with its discrete l_w^2 norm of the coefficients $(u_k^l)_{(k,l)} \in \mathbb{R}$, i.e.

$$\sqrt{|||u_k^l|||_w^2} := \sum_l \sum_k w^2(2^{-l}k)|u_k^l|^2.$$

Theorem 3.3 *Let us assume that Assumptions 3.1 and 3.2 are valid. For any function* $u = \sum_{l=l_0}^{\infty} \sum_k u_k^l \psi_k^l \in L_w^2\,((0,1))$ *holds*

$$\| u \|_w^2 \approx \sqrt{|||u_k^l|||_w^2}.$$

Proof. From Theorem 3.2 we conclude

$$\begin{aligned}\| u \|_w^2 &= \sum_{l,l'} \sum_{k,k'} u_k^l u_{k'}^{l'} w(2^{-l}k) w(2^{l'}k') \left(\psi_k^l, \psi_{k'}^{l'}\right)_w \\ &\leq \| M \|_2 \left(\sum_l \sum_k \left|u_k^l\right| w(2^{-l}k)\right)^2 \leq \sqrt{c|||u_k^l|||_w^2}.\end{aligned}$$

To prove the lower estimate we consider the dual system

$$\widetilde{v} = \sum_l \sum_k \widetilde{v}_k^l \widetilde{\psi}_k^l = G(\widetilde{v}_k^l)$$

in the dual space $L_{w^{-1}}^2\,((0,1))$. We denote by $\widetilde{M}$ the mass matrix of the dual wavelet basis $\widetilde{\psi}_k^l$ with respect to the $L_{w^{-1}}^2\,((0,1))$ innerproduct. Then, by the same arguments

$$\| \widetilde{v} \|_{w^{-1}}^2 \leq \| \widetilde{M} \|_2 \sqrt{|||\widetilde{v}_k^l|||_{w^{-1}}^2}.$$

This means $G : l_{w^{-1}}^2 \to L_{w^{-1}}^2\,((0,1))$ is bounded. Therefore, the adjoint operator $G^* : L_w^2\,((0,1)) \to l_w^2$ is bounded, too. G^* is explicitly given by

$$G^* u := \left(\langle u, \widetilde{\psi}_k^l \rangle\right)_{l,k} = (u_k^l)_{l,k}$$

which proves the lower bound. □

Remark 3.5 The presented result is simliar to the result of Zhang, [33]. It would be an interesting question to charcterize the operators (bilinear forms) for which a diagonal preconditioning in wavelet bases holds. To our knowledge we would like to mention that we are not aware about any result concerning this question.

4 Application to the p-Version of the FEM

The theory of Chapter 3 can be applied to find a fast solver for the element stiffness matrices in the p-Version of the FEM in two and three dimensions. As indicated in the introduction, we precondition the p-FEM stiffness matrices by corresponding h-FEM matrices which are spectrally equivalent and for which efficient inversion is possible. Previous work focused on tensor products of linear elements on suitably graded meshes, see Ivanov and Korneev [18], [19], Jensen and Korneev [20], and the pioneering work by Mund [15].

4.1 Model problem

We consider the model problem

$$-\Delta u = f \quad \text{in} \quad \mathcal{R} = (-1,1)^{\hat{d}}, \quad \hat{d} = 2, 3 \tag{4.1}$$

$$u = 0 \quad \text{on} \quad \partial\mathcal{R}. \tag{4.2}$$

We solve (4.1, 4.2) approximately using the $p-$version of the FEM with only one element $\mathcal{R}$. As finite element space, we choose $\mathbb{M} = \{u\ |_{\mathcal{R}} \in Q^p, u = 0 \text{ on } \partial\mathcal{R}\}$, where Q^p is the space of all polynomials of degree p in each variable. The discretized problem is: find $u_p \in \mathbb{M}$

$$\int_{\mathcal{R}} \nabla u_p \cdot \nabla v_p \,\mathrm{d}(x,y) = \int_{\mathcal{R}} f v_p \,\mathrm{d}(x,y)$$

for all $v_p \in \mathbb{M}$. As basis in $\mathbb{M}$, we choose the integrated Legendre polynomials, which we define below.

Let for $i = 0, 1, \ldots, L_i(x) = \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}(x^2-1)^i$ for $i \geq 2$ the i-th Legendre polynomial,

$$\hat{L}_i(x) = \sqrt{\frac{(2i-3)(2i-1)(2i+1)}{4}} \int_{-1}^{x} L_{i-1}(s)\,\mathrm{d}s$$

the i-th integrated Legendre polynomial. $\hat{L}_0(x) = \frac{1+x}{2}$, $\hat{L}_1(x) = \frac{1-x}{2}$. These scaled integrated Legendre polynomials were introduced by Jensen and Korneev [20]. As basis in $\mathbb{M}$, we choose

$$\hat{L}_{ij}(x,y) = \hat{L}_i(x)\hat{L}_j(y), \quad \text{or} \quad \hat{L}_{ijk}(x,y,z) = \hat{L}_i(x)\hat{L}_j(y)\hat{L}_k(z), \tag{4.3}$$

with $2 \leq i, j, k \leq p$ for $\hat{d} = 2$ or $\hat{d} = 3$.

In order to satisfy (4.2), the polynomials $\hat{L}_0$ and $\hat{L}_1$ are omitted. The stiffness matrix $K_{\hat{d}}$ for (4.1) with $\hat{d} = 2$ is determined by $K_2 = (a_{ij,kl})_{i,j=2;k,l=2}^{p}$, where

$$a_{ij,kl} = \int_{\mathcal{R}} \nabla \hat{L}_{ij}(x, y) \cdot \nabla \hat{L}_{kl}(x, y) \, \mathrm{d}(x, y) \quad \text{for} \quad \hat{d} = 2.$$

By a simple calculation it follows $K_2 = F \otimes N + N \otimes F$ for $\hat{d} = 2$ and $K_3 = F \otimes F \otimes N + F \otimes N \otimes F + N \otimes F \otimes F$ for $\hat{d} = 3$, where

$$F = \begin{pmatrix} 1 & 0 & -c_2 & 0 & \cdots \\ & 1 & 0 & -c_3 & \ddots \\ & & 1 & 0 & \ddots \\ & \text{SYM} & \ddots & \ddots & \ddots \\ & & & & 1 \end{pmatrix}$$

is the one-dimensional mass-matrix and $N = \mathrm{diag}(d_i)_{i=2}^{p}$ is the one-dimensional stiffness matrix with the coefficients $c_i = \sqrt{\frac{(2i-3)(2i+5)}{(2i-1)(2i+3)}}$, and $d_i = \frac{(2i-3)(2i+1)}{2}$, [20]. Using a permutation P of rows and columns, there holds

$$P^t F P = \begin{pmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{pmatrix}, \quad P^t N P = \begin{pmatrix} N_1 & \mathbf{0} \\ \mathbf{0} & N_2 \end{pmatrix}$$

where $N_1 = \mathrm{diag}(d_2, d_4, d_6, \ldots)$, $N_2 = \mathrm{diag}(d_3, d_5, d_7, \ldots)$,

$$F_1 = \mathrm{tridiag}(-\mathbf{c}_e, \mathbf{1}, -\mathbf{c}_e), \quad F_2 = \mathrm{tridiag}(-\mathbf{c}_o, \mathbf{1}, -\mathbf{c}_o)$$

with $\mathbf{c}_e = (c_2, c_4, c_6, \ldots)$ and $\mathbf{c}_o = (c_3, c_5, c_7, \ldots)$.

4.2 Preconditioning

We introduce now the following two matrices T and $\hat{M}$, given by

$$T = \mathrm{tridiag}(-\mathbf{1}, \mathbf{2}, -\mathbf{1}) \quad \text{and} \quad \hat{M} = \mathrm{tridiag}(\mathbf{a}, \mathbf{b}, \mathbf{a}), \tag{4.4}$$

where $\mathbf{a} = \left(i^2 + i + \frac{3}{10}\right)_{i=1}^{n-1}$ and $\mathbf{b} = \left(4i^2 + \frac{2}{5}\right)_{i=1}^{n}$.

These matrices can be used as preconditioniers for the matrices F and N. The following lemma holds, (cf. [1] and the references therein to Jensen and Korneev [20]).

Lemma 4.1 *The following eigenvalue estimates are valid for $i = 1, 2$*

$$\lambda_{min}(N_i^{-\frac{1}{2}} \hat{M} N_i^{-\frac{1}{2}}) \geq c, \ \ \lambda_{max}(N_i^{-\frac{1}{2}} \hat{M} N_i^{-\frac{1}{2}}) \leq C,$$
$$\lambda_{min}(F_i^{-\frac{1}{2}} T F_i^{-\frac{1}{2}}) \geq \frac{c}{1 + \log n}, \ \ \lambda_{max}(F_i^{-\frac{1}{2}} T F_i^{-\frac{1}{2}}) \leq C.$$

Now, we show how the matrices T and $\hat{M}$ arise. To this end, we consider the following auxiliary problem in one dimension: find $u \in H_0^1((0,1))$, such that

$$a_1(u,v) = a_s(u,v) + a_m(u,v) = \langle g, v\rangle \tag{4.5}$$

holds for all $v \in H_0^1((0,1))$. The bilinear forms $a_s(\cdot,\cdot)$ and $a_m(\cdot,\cdot)$ are defined as follows

$$a_s(u,v) = \int_0^1 u'(\xi)v'(\xi)\,\mathrm{d}\xi = \langle u', v'\rangle_{w=1} \quad \forall u,v \in H_0^1((0,1)),$$
$$a_m(u,v) = \int_0^1 \xi^2 u(\xi)v(\xi)\,\mathrm{d}\xi = \langle u, v\rangle_{w=\xi} \quad \forall u,v \in L_w^2((0,1)).$$

We discretize this one-dimensional problem (4.5) by using linear elements on the uniform mesh $\bigcup_{i=0}^{n-1}\tau_i^l$, where $\tau_i^l = \left(\frac{i}{n}, \frac{i+1}{n}\right)$. The number n of elements is assumed to be a power of two, i.e. $n = 2^l$ where l denotes the level number. On this uniform mesh we introduce the standard one-dimensional hat-functions $\phi_i^{(1,l)}$ for $i = 1, \dots, n-1$. Let

$$(T_w)_{ij} = \langle(\phi_i^{(1,l)})', (\phi_j^{(1,l)})'\rangle_w \quad \text{and} \quad (M_w)_{ij} = \langle\phi_i^{(1,l)}, \phi_j^{(1,l)}\rangle_w. \tag{4.6}$$

Then, an easy calculation shows, cf. [1], $T_1 = \frac{n}{2}T$ and $M_\xi = c\hat{M}$ with some constant c depending on n, where a subscript ξ denotes the weight function $w(\xi) = \xi$ and a subscript 1 denotes unweighted the inner product.

So, we see the reason for introducing the matrices T and $\hat{M}$ (4.4). By tensor product arguments, the following theorem holds.

Theorem 4.1 *Let $A_2 = T \otimes \hat{M} + \hat{M} \otimes T$ and $A_3 = T \otimes T \otimes \hat{M} + T \otimes \hat{M} \otimes T + \hat{M} \otimes T \otimes T$. Furthermore let*

$$\widetilde{K}_{\hat{d}} = P_{\hat{d}}\mathrm{blockdiag}\left[A_{\hat{d}}\right]_{i=1}^{2^{\hat{d}}} P_{\hat{d}}^t \quad \text{for} \quad \hat{d} = 2, 3,$$

where P_2 and P_3 are explicitely given permutation matrices. Then the condition number κ of $\widetilde{K}_{\hat{d}}^{-\frac{1}{2}} K_{\hat{d}} \widetilde{K}_{\hat{d}}^{-\frac{1}{2}}$ can be estimated by

$$\kappa(\widetilde{K}_{\hat{d}}^{-\frac{1}{2}} K_{\hat{d}} \widetilde{K}_{\hat{d}}^{-\frac{1}{2}}) \le c(1+\log p)^{\hat{d}-1} \quad \text{for} \quad \hat{d} = 2, 3.$$

Proof. The assertion follows by Lemma 4.1 and tensor product arguments. For more details see [1]. □

4.3 Wavelet preconditioning

The matrices A_2 and A_3 are the stiffness matrices for discretizing in $\Omega = (0,1)^{\hat{d}}$ the following singular elliptic problems

$$
\begin{aligned}
-x^2 u_{yy} - y^2 u_{xx} &= f,\ u\,|_{\partial\Omega} = 0 \quad \text{for} \quad \hat{d} = 2,\\
x^2 u_{yyzz} + y^2 u_{xxzz} + z^2 u_{xxyy} &= f,\ u\,|_{\partial\Omega} = 0 \quad \text{for} \quad \hat{d} = 3
\end{aligned}
$$

using bi- or trilinear finite elements on the graded tensor product mesh $\tau_i^l \times \tau_j^l$ for $\hat{d} = 2$ or $\tau_i^l \times \tau_j^l \times \tau_k^l$ for $\hat{d} = 3$. For more details, see [1].

Using Theorem 3.3 and Theorem 4.1 a wavelet preconditioner for $K_{\hat{d}}$ can therefore be built as follows.

Let Q be the basis transformation matrix from the wavelet basis $\{\psi_k^l\}_{k,l}$ to the basis $\{\phi_i^{(1,l)}\}_{i=1}^{2^l-1}$. Define the mass matrix and stiffness matrix in the wavelet basis, $D_{m,w} = \operatorname{diag}\left(\langle \psi_k^l, \psi_k^l \rangle_w\right)$, $D_{s,w} = \operatorname{diag}\left(\langle (\psi_k^l)', (\psi_k^l)' \rangle_w\right)$. From Theorem 3.3 with $w(\xi) = \xi$ and from the properties of a multi resolution basis, cf. (2.9), we have

$$\kappa(Q^t D_{m,\xi}^{-1} Q \hat{M}) \le c \quad \text{and} \quad \kappa(Q^t D_{s,1}^{-1} Q T) \le c$$

for some $c > 0$ independent of p. Thus, from the properties of the Kronecker product follows $\kappa(Q_2 A_2) \le c$ where

$$Q_2 = (Q^t \otimes Q^t)(D_{m,\xi} \otimes D_{s,1} + D_{s,1} \otimes D_{m,\xi})^{-1}(Q \otimes Q) \tag{4.7}$$

and by Theorem 4.1, $\kappa\left(P_2 \operatorname{blockdiag}\,[Q_2]_{i=1}^4\, P_2^t K_2\right) \le c(1 + \log p)$. Defining a matrix

$$
\begin{aligned}
Q_3 = {} & (Q^t \otimes Q^t \otimes Q^t)(D_{m,\xi} \otimes D_{s,1} \otimes D_{s,1}\\
& + D_{s,1} \otimes D_{m,\xi} \otimes D_{s,1} + D_{s,1} \otimes D_{s,1} \otimes D_{m,\xi})^{-1}(Q \otimes Q \otimes Q)
\end{aligned} \tag{4.8}
$$

a similar holds for $\hat{d} = 3$.

Theorem 4.2 *Let us assume that Assumptions 3.1, 3.2 with* $\alpha = 1$ *and relation* (2.9) *for* $t = 1$ *are satisfied. Then, the matrices* $Q_{\hat{d}}$ (4.7) *and* (4.8) *satisfy*

$$\kappa\left(P_{\hat{d}} \operatorname{blockdiag}\left[Q_{\hat{d}}\right]_{i=1}^{2^{\hat{d}}} P_{\hat{d}}^t K_{\hat{d}}\right) \le c(1 + \log p)^{\hat{d}-1} \quad \text{for } \hat{d} = 2, 3.$$

Therefore, a nearly optimal preconditioner for the element stiffness matrix $K_{\hat{d}}$ in the p-version of the FEM is found.

Remark 4.1 This approach can be extended to discretizations of (4.1), (4.2), in which the polynomial degree in the variables x and y is anisotropic. If $\mathcal{R} = (-a_1, a_1) \times (-a_2, a_2)$ or $\mathcal{R} = (-a_1, a_1) \times (-a_2, a_2) \times (-a_3, a_3)$ the preconditioners $Q_{\hat{d}}$ can be used, too. However, instead of (4.7),

$$Q_2 = (Q^t \otimes Q^t)(\frac{a_1}{a_2} D_{m,\xi} \otimes D_{s,1} + \frac{a_2}{a_1} D_{s,1} \otimes D_{m,\xi})^{-1}(Q \otimes Q)$$

should be used. Then, Theorem 4.2 holds with constants independent of the parameters a_1 and a_2. An analogous modification is possible for Q_3 (4.8).

4.4 Numerical results

We now illustrate the performance of the wavelet preconditioner by numerical examples. We consider exemplarily the following three frequently used multiresolution bases $\psi_{2,s}$, $s = 2, 4, 6$, cf. Figure 1. The functions $\psi_{2,s}$ are piecewise linear and satisfy (2.3) with $d + 1 = 2$ and (2.2) with $\widetilde{d} + 1 = s$, $s = 2, 4, 6$. Note that $\widetilde{\psi}_{22}$ is not continuous. For more details about the wavelet basis we refer to [13].

4.4.1 Condition number of mass matrix Figure 2 displays the condition numbers of the matrix M (3.1) with the scaling function $w(\xi) = \xi$ in the multiresolution bases $\psi_{2,s}$, $s = 2, 4, 6$. Note that the entry corresponding to

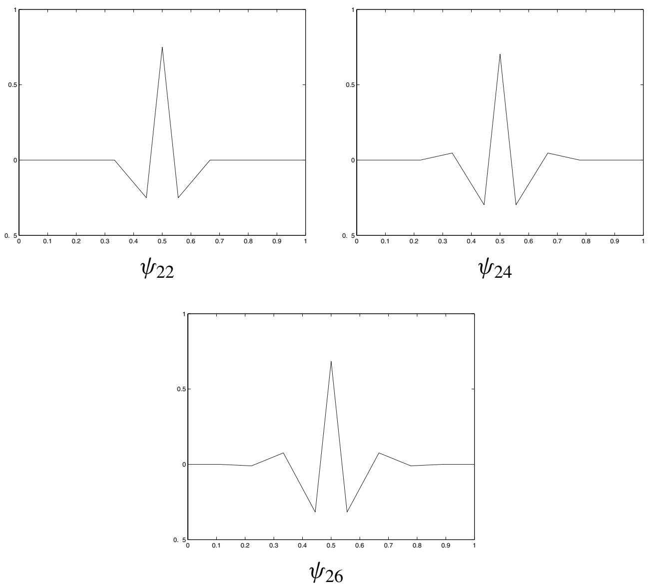

Fig. 1. Wavelets ψ_{22}, ψ_{24}, ψ_{26}

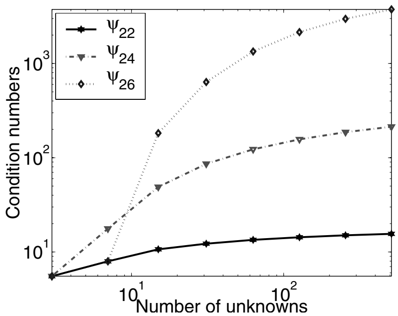

Fig. 2. Condition number of the mass matrix

ψ_k^l is scaled with $w(2^{-l}k)^2$. With an another choice of diagonal scaling the condition number cannot be significantly improved in the case of $w(\xi) = \xi$.

From the results it can be concluded that the condition numbers depend strongly on the choice of the wavelet. The condition numbers appear to grow at worst proportionally to the logarithm of the number of unknowns for all multiresolution bases considered with large differences in the actual values. Wavelet ψ_{22} (not covered by our results) shows the lowest condition numbers.

4.4.2 Preconditioner for the p-Version FEM In this subsection, the system $K_{\hat{d}}\underline{u} = \underline{f}$ for $\hat{d} = 2, 3$ is considered. In all numerical examples, the number of iterations of the pcg-method for reducing the error of the residuum in the preconditioned energy norm to the factor $\varepsilon = 10^{-10}$ is displayed. The matrices $Q_{\hat{d}}$, (4.7) for $\hat{d} = 2$ and (4.8) for $\hat{d} = 3$, are chosen as preconditioner. Figure 1 displays the number of iterations for $\hat{d} = 2, 3$.

In both cases, the number of iterations grows moderately for the wavelet ψ_{22}. However, for ψ_{26} the growth is logarithmic, but the absolute number of iterations, i.e. about 1000 for $\hat{d} = 3$ and $p = 255$, are too large.

Table 1. Number of iterations of the pcg for $K_{\hat{d}}$ with prec. $Q_{\hat{d}}$, $\hat{d} = 2$ (above), $\hat{d} = 3$ (bottom)

p	3	7	15	31	63	127	255	511	1023	2047	4095
ψ_{22}	2	3	24	33	40	46	52	56	61	65	69
ψ_{24}	2	3	24	41	59	89	123	162	195	220	246
ψ_{26}	2	3	24	41	78	150	309	548	819	1102	1389

p	3	7	15	31	63	127	255
ψ_{22}	2	3	45	55	62	72	84
ψ_{24}	2	3	45	75	112	179	252
ψ_{26}	2	3	45	75	177	483	1082

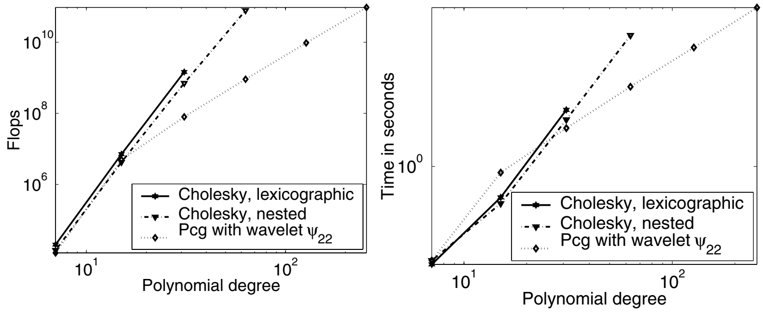

Fig. 3. Comparison of direct and indirect methods for $K_3\underline{u} = \underline{f}$

Now, we compare these iterative methods with direct solvers for $K_3\underline{u} = \underline{f}$. Two direct methods are considered:

- Cholesky-decomposition with lexicographic ordering of the unknowns,
- Cholesky-decomposition with a nested ordering of the unknowns, cf. [16], [17].

Both methods are compared with a pcg-method using the preconditioner Q_3, (4.8) and the wavelet ψ_{22}. The relative accuracy is $\varepsilon = 10^{-10}$. On the left picture of Figure 3, the number of floating point operations are compared, on the right one the time for solving $K_3\underline{u} = \underline{f}$. From the results can be concluded, that for $p \leq 15$ the nested Cholesky decomposition is faster than the pcg-method with wavelet-preconditioner. However, for $p > 15$ the iterative solver is faster.

We observe also that for $\hat{d} = 2$ the preconditioner based on ψ_{22} compares favourably with algebraic multigrid preconditioners developed in [2], Table 4.3.

5 Application to degenerate elliptic problems

Second order elliptic problems with degenerate diffusion arise in a number of applications. We mention here only axisymmetric problems in three dimensions and the pricing of contracts on assets driven by Brownian motion with stochastic volatility (see, e.g., [27]). The weighted norm equivalences established in this paper allow us to precondition finite element discetizations of such equations optimally. To our knowledge preconditioning of degenerate diffusion coefficients is considered only in few papers.

There exists some papers about diffusion coefficients with jumps, [3], [26]. In [21], a preconditioner for a degenerate problem is proposed by considering a problem with jumping diffusion coeffcients. It is conceivable to extend these results to the present problems.

5.1 1-d Model problem

We consider the following model problem in the one-dimensional domain $\Omega = (0, 1)$: find $u \in H^1_{w,0}(\Omega)$ such that

$$a(u, v) := \langle u', v'\rangle_w + \langle u, v\rangle = \int_0^1 \left(\xi^2 u'v' + uv\right)\, \mathrm{d}\xi$$

$$= \int_0^1 f v \,\mathrm{d}\xi \quad \forall v \in H^1_{w,0}(\Omega) \tag{5.1}$$

where $H^1_{w,0}(\Omega)$ denotes the H^1 space with weight $w(\xi) = \xi$, i.e.

$$H^1_{w,0}(\Omega) = \{u \in L^2((0, 1)) : \xi u' \in L^2((0, 1)), u(1) = 0\}.$$

The space $H^1_{w,0}(\Omega)$ equipped with the norm $\|u\|^2_{1,w} := a(u, u)$ is a Hilbert space and hence the problem (5.1) admits, for every $f \in (H^1_{w,0}(\Omega))^*$, a unique solution by the Lax-Milgram Lemma.

We discretize (5.1) by piecewise linear finite elements on a uniform mesh of meshwidth $h = 2^{-L}$, $L \geq 1$, with zero Dirichlet boundary conditions at the right end point $x = 1$. Denoting by $\mathbb{V}^0_L \subset H^1_{w,0}(\Omega)$ the corresponding subspace and, as in the case of $H^1_0((0, 1))$, we denote the corresponding spline wavelet spaces by $\mathbb{W}^0_l$, $l = 0, ..., L$ and the wavelet bases by $\{\psi^l_k\}$, again normalized so that

$$||\psi^l_k||_{L^2(\Omega)} = 1. \tag{5.2}$$

The stiffness matrix A corresponding to the form $a(\cdot, \cdot)$ is then given by

$$A = D_\xi + G_1, \tag{5.3}$$

where

$$D_w = \left(\langle(\psi^l_k)', (\psi^{l'}_{k'})'\rangle_w\right), \quad G_w = \left(\langle\psi^l_k, \psi^{l'}_{k'}\rangle_w\right). \tag{5.4}$$

Due to the normalization (5.2), we have a norm equivalence analogous to (2.9)

$$\|u\|^2_t \sim \sum_{l=-1}^{\infty} 2^{2lt} \sum_{k\in\nabla_l} |u_{l,k}|^2 \tag{5.5}$$

for all $u \in H^1_0(\Omega)$ and for $t \in (-\widetilde{\gamma}_0, \gamma_0)$, where $u_{l,k} = \langle u, \widetilde{\psi}^l_k\rangle$. Analogous to Theorem 3.3 we can prove

Theorem 5.1 *Suppose that Assumptions 3.1 and 3.2 are satisfied for the bases* $\{(\psi_k^l)'\}$ *and* $\{(\widetilde{\psi}_k^l)'\}$. *Assume further that relation* (5.5) *holds with* $t = 0$, *i.e. that* $\{\psi_k^l\}$ *is a Riesz basis. Let* $\gamma_0 > 1$.
Then, for $u = \sum_{l=l_0}^{L} \sum_k u_k^l \psi_k^l$ *holds the norm equivalence*

$$\| u' \|_w^2 \approx \sum_l 2^{2l} \sum_k w^2(2^{-l}k)|u_k^l|^2 = \sum_l \sum_k k^2 |u_k^l|^2$$

uniformly in L.

Note that the summation over k runs, in level l, from $k = 1$ to $k_{max} = O(2^l)$, i.e. the weight in the discrete norm equivalence ranges from L^2 for the contributions near $x = 0$ to H^1 near $x = 1$.

As a corollary, we obtain a preconditioner for the matrix A in (5.1) where $w(\xi) = \xi$.

Proposition 5.1 *Denote by C the matrix with entries given by*

$$C_{(l,k),(l',k')} = k\delta_{k,k'}\delta_{l,l'}.$$

Then there is $c > 0$ *independent of L such that for the stiffness matrix A of* (5.1) *holds*

$$\mathrm{cond}_2(C^{-1}AC^{-1}) \le c < \infty.$$

5.2 2-d selfadjoint anisotropic problems

Here, we consider diffusion problems with coefficients which degenerate at the boundary. They are models for the pricing of contracts on assets driven by Brownian motion with stochastic volatility, as e.g. in [27]. Note particularly that these differential equations from finance are parabolic with degenerate elliptic operator. The singular weight function in these applications is always a tensor product of univariate singular weights. Various singular weight functions appear in practice, depending on the particular stochastic volatility model. Rather than giving a detailed presentation of these models (containing numerous parameters and lower order differential operators), we show in the following how our univariate preconditioning results extend readily to the higher dimensional case.

We consider exemplarily the following two problems with degenerate coefficients in the two-dimensional domain $\Omega = (0, 1)^2$.

- find $u \in H^1_{w,0}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega (w^2(x)w^2(y)u_x v_x + u_y v_y + uv)\, \mathrm{d}(x, y) \\ &= \int_\Omega f v\, \mathrm{d}(x, y) \quad \forall v \in H^1_{w,0}(\Omega) \end{aligned} \tag{5.6}$$

- find $u \in H^1_{w,w,0}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega (w^2(x)w^2(y)(u_x v_x + u_y v_y) + uv)\,\mathrm{d}(x,y)\\ &= \int_\Omega f v\,\mathrm{d}(x,y) \quad \forall v \in H^1_{w,w,0}(\Omega)\end{aligned} \tag{5.7}$$

where $H^1_{w,0}(\Omega)$ denotes a weighted H^1 space, i.e.

$$\begin{aligned}H^1_{w,0}(\Omega) = \{u \in L^2(\Omega),\, u_y,\, w(x)w(y)u_x \in L^2(\Omega),\\ u(x,1) = u(1,y) = 0\}\end{aligned}$$

and $H^1_{w,w,0}(\Omega)$ is the weighted Sobolev space

$$\begin{aligned}H^1_{w,w,0}(\Omega) = \{u \in L^2(\Omega),\, w(x)w(y)u_x,\, w(x)w(y)u_y \in L^2(\Omega),\\ u(x,1) = u(1,y) = 0\}.\end{aligned}$$

We discretize (5.6), (5.7) by piecewise bilinear finite elements on the uniform tensor product mesh $\tau_i^l \times \tau_j^l$. The stiffness matrix in the wavelet basis $\{\psi_k^l(x)\psi_{k'}^{l'}(y)\}_{(k,l),(k',l')}$ is given by

$$\begin{aligned}B_2 &= D_\xi \otimes G_\xi + G_1 \otimes D_1 + G_1 \otimes G_1 \quad \text{for (5.6)},\\ B_3 &= D_\xi \otimes G_\xi + G_\xi \otimes D_\xi + G_1 \otimes G_1 \quad \text{for (5.7)}\end{aligned}$$

with the matrices D_w and G_w introduced by relation (5.4). Denote by $C_{s,w}$ and $C_{m,w}$ the diagonal matrices with entries given by

$$\begin{aligned}\left(C_{s,w}\right)_{(l,k),(l',k')} &= \delta_{k,k'}\delta_{l,l'}2^{2l}w^2(2^{-l}k),\\ \left(C_{m,w}\right)_{(l,k),(l',k')} &= \delta_{k,k'}\delta_{l,l'}w^2(2^{-l}k)\end{aligned}$$

and let

$$\begin{aligned}C_2 &= (C_{s,\xi} \otimes C_{m,\xi} + C_{m,1} \otimes C_{s,1} + C_{m,1} \otimes C_{m,1})^{\frac{1}{2}},\\ C_3 &= (C_{s,\xi} \otimes C_{m,\xi} + C_{m,\xi} \otimes C_{s,\xi} + C_{m,1} \otimes C_{m,1})^{\frac{1}{2}}.\end{aligned}$$

Then, by Theorem 5.1, Theorem 3.3, relation (2.9) and tensor product arguments we find

Theorem 5.2 *There holds for $i = 2, 3$, $\mathrm{cond}_2(C_i^{-1}B_iC_i^{-1}) \le c < \infty$ where the constant c is independent of the level number L.*

We give now numerical experiments for the condition number of $C^{-1}AC^{-1}$ in the l_2-norm for the wavelets ψ_{22}. Note that this wavelet does not satisfy the assumptions of Theorem 5.1. Unlike in the one-dimensional case, there are now several ways to extract a preconditioner from the stiffness matrix A. We compare here numerically three different constructions of preconditioners C.

Cases I and III correspond to the usual block-diagonal preconditioners similar to those employed in one dimension. The numerical experiments revealed that although the condition number is bounded uniformly in the number of levels L, its absolute value is still rather large. In the construction of the preconditioner, the most delicate problem are the wavelets at the boundary $x = 0$. For improving the condition number of $C^{-1}AC^{-1}$ we consider therefore as case II a matrix C^{II} in which the entries corresponding to wavelets ψ_k^l with $0 \in \operatorname{supp} \psi_k^l$, i.e. with $k = 1$, are not set to 0. Then, for solving $C^{\mathrm{II}}\underline{w} = \underline{r}$ a linear system of dimension $\log_2 n$ has to be solved via Cholesky decomposition. Specifically, below the following three types of preconditioning matrices C are considered.

- case I:

$$C^{\mathrm{I}}_{(l,k),(l',k')} = \sqrt{\langle(\psi_k^l)', (\psi_k^l)'\rangle_w}\,\delta_{k,k'}\delta_{l,l'},$$

- case II:

$$(C^{\mathrm{II}}_{(l,k),(l',k')})^2 = \begin{cases} \langle(\psi_k^l)', (\psi_{k'}^{l'})'\rangle_w & \text{if} \quad k = k', l = l' \\ \langle(\psi_k^l)', (\psi_{k'}^{l'})'\rangle_w & \text{if} \quad k = k' = 1 \\ 0 & \qquad \text{else} \end{cases},$$

- case III:

$$C^{\mathrm{III}}_{(l,k),(l',k')} = k\delta_{k,k'}\delta_{l,l'}.$$

Figure 4 displays the condition numbers of $C^{-1}AC^{-1}$ choosing the wavelets ψ_{22}. One can see in all cases the same asymptotic behaviour. However, the

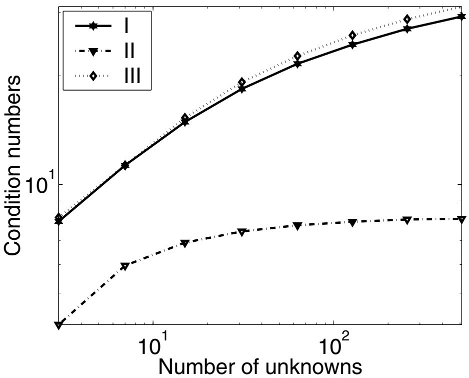

Fig. 4. Condition number of the matrix A

condition number is about 8 for the case C^{II}, in contrast to about 30 for the other cases.

Next, we consider the matrices $C_i^{-1} B_i C_i^{-1}$. In the corresponding one dimensional example, we have seen that the matrix $C = C^{\mathrm{II}}$ reduces the condition number of $C^{-1}AC^{-1}$ in comparison to diagonal matrices $C = C^{\mathrm{I}}$ or $C = C^{\mathrm{III}}$. Thus, instead of $C_i^{-1} B_i C_i^{-1}$, $i = 2, 3$ we consider $(C_i^{\mathrm{II}})^{-1} B_i$ where

$$C_2^{\mathrm{II}} = C^{\mathrm{II}} \otimes C_{m,\xi} + C_{m,1} \otimes C_{s,1} + C_{m,1} \otimes C_{m,1},$$
$$C_3^{\mathrm{II}} = C^{\mathrm{II}} \otimes C_{m,\xi} + C_{m,\xi} \otimes C^{\mathrm{II}} + C_{m,1} \otimes C_{m,1}.$$

Note, that the matrices $C_{m,1}$, $C_{s,1}$ and $C_{m,\xi}$ are diagonal matrices. Moreover, the matrix C^{II} can be written as

$$C^{\mathrm{II}} = \begin{pmatrix} D^{\mathrm{II}} & \mathbf{0} \\ \mathbf{0} & R^{\mathrm{II}} \end{pmatrix}$$

where D^{II} is a diagonal matrix and R^{II} is a fully populated matrix of dimension $\log_2 n$, corresponding to the wavelets with $k = 1$. Thus, for solving the $n^2 \times n^2$ system $C_2^{\mathrm{II}} \underline{w} = \underline{r}$ we have to solve n symmetric, positive definite linear systems of dimension $\log_2 n$ and a diagonal system of dimension $n^2 - n \log_2 n$. Using here a Cholesky decomposition, the total cost for these solves is asymptotically $n^2 + \frac{1}{6} n (\log_2 n)^3$. With analogous arguments it can be shown that the total cost for solving $C_3^{\mathrm{II}} \underline{w} = \underline{r}$ is asymptotically $n^2 + \frac{1}{6}(2n-1)(\log_2 n)^3 + \frac{1}{6}(\log_2 n)^6$. Table 2 displays the condition numbers of $(C_i^{\mathrm{II}})^{-\frac{1}{2}} B_i (C_i^{\mathrm{II}})^{-\frac{1}{2}}$ for $i = 2, 3$ in the l_2-norm using the wavelets ψ_{22}. We

Table 2. Condition numbers of $(C_i^{\mathrm{II}})^{-\frac{1}{2}} B_i (C_i^{\mathrm{II}})^{-\frac{1}{2}}$

Level	3	4	5	6
$\mathrm{cond}_2((C_2^{\mathrm{II}})^{-\frac{1}{2}} B_2 (C_2^{\mathrm{II}})^{-\frac{1}{2}})$	9.9	12.1	13.9	15.4
$\mathrm{cond}_2((C_3^{\mathrm{II}})^{-\frac{1}{2}} B_3 (C_3^{\mathrm{II}})^{-\frac{1}{2}})$	6.0	11.3	15.7	19.8

observe moderate growth of the condition numbers with respect to small n.

5.3 Nonselfadjoint degenerate problem

We consider a degenerate parabolic problem. In $\Omega = (0, 2) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, $Q_T = \Omega \times (0, T)$, $T > 0$ and $\Sigma_T = \partial\Omega \times (0, T)$ we solve the parabolic differential equation

$$
\begin{aligned}
u_t - \frac{1}{2}x^2|y|^2 u_{xx} - \frac{1}{2}\beta^2 u_{yy} - \rho\beta x|y|u_{yx}& \\
-r\big(xu_x - u\big) - \alpha(m-y)u_y &= g \quad \text{in } Q_T \\
u &= 0 \quad \text{on } \Sigma_T \\
u(\cdot,0) &= u_0 \text{ in } \Omega,
\end{aligned}
$$

which arises in option pricing for stochastic volatility models [27]. In this example, the elliptic part of the operator degenerates at $y = 0$, which is in the interior of the domain Ω. We consider the constants $\alpha = 1$, $\beta = \frac{1}{\sqrt{2}}$, $\rho = -0.5$, $r = 0.05$ and $m = 0.2$ and cast the problem in variational form: given $g \in V^*$, find $u \in L^2(0,T;V)$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(u,v)_{L^2(\Omega)} + a(u,v) = \langle g, v\rangle_{V^*\times V} \qquad \forall v \in V, \\
&u(0,\cdot) = 0,
\end{aligned}
\tag{5.8}
$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$
\begin{aligned}
a(u,v) = \frac{1}{2}\int_\Omega & x^2 w^2(y)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\mathrm{d}x\mathrm{d}y \\
&+ \int_\Omega x w^2(y)\frac{\partial u}{\partial x} v\mathrm{d}x\mathrm{d}y + \frac{1}{2}\beta^2\int_\Omega \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\mathrm{d}x\mathrm{d}y \\
&+\rho\beta\int_\Omega \left(xw(y)\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + w(y)\frac{\partial u}{\partial y}v\right)\mathrm{d}x\mathrm{d}y - r\int_\Omega x\frac{\partial u}{\partial x}v\mathrm{d}x\mathrm{d}y \\
&+\alpha\int_\Omega (y-m)\frac{\partial u}{\partial y}v\mathrm{d}x\mathrm{d}y + r\int_\Omega uv\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

with $w(y) = |y|$ (various other, singular $w(y)$ could be chosen depending on the volatility model). In (5.8) the time derivative is taken in the sense of distributions and V denotes the weighted Sobolev space

$$
V = \left\{ v \;\middle|\; \left(v, \frac{\partial v}{\partial y}, xw(y)\frac{\partial v}{\partial x}\right) \in \big(L^2(\Omega)\big)^3 \right\}. \tag{5.9}
$$

which is equipped with the norm

$$
\|v\|_V^2 = \int_\Omega \left(v^2 + \left(\frac{\partial v}{\partial y}\right)^2 + x^2 w^2(y)\left(\frac{\partial v}{\partial x}\right)^2\right)\mathrm{d}x\mathrm{d}y \tag{5.10}
$$

By V^* we denote the dual of V with respect to the pivot space $L^2(\Omega)$.

We discretize (5.8) in time by the θ-scheme with time step $k = 10^{-2}$ and in Ω by tensor product wavelets with levels $\mathbf{L} = (L+1, L)$. The stiffness matrix of the form $a(\cdot,\cdot)$ is, with the univariate advection matrix

$$
B_w = \Big(\langle(\psi_k^l)', \psi_{k'}^{l'}\rangle_w\Big)_{(k,l),(k',l')},
$$

given by (with $B_{\sqrt{\xi}} + B^t_{\sqrt{\xi}} + M_1 = 0$)

$$\mathbf{K} := \frac{1}{2}\theta D^{(x)}_{\xi} \otimes G^{(y)}_{|\xi|} + \theta B^{(x)}_{\sqrt{\xi}} \otimes \Big(G^{(y)}_{|\xi|} - rG^{(y)}_1 - \beta\rho B^{(y)}_{\sqrt{|\xi|}}\Big) \\ +\theta G^{(x)}_1 \otimes \Big(\big(1/(k\theta) + r\big)G^{(y)}_1 + \frac{1}{2}\beta^2 D^{(y)}_1 + \alpha B^{(y)}_{\sqrt{\xi - m}}\Big).$$

We only consider the preconditioner of type II

$$\big(\mathbf{C}^{\mathrm{II}}\big)^2 := \frac{\theta}{2} C^{\mathrm{II}} \otimes C_{m,|\xi|} + \theta C_{m,1} \otimes \Big(\big(1/(k\theta) + r\big)C_{m,1} + \frac{1}{2}\beta^2 C_{s,1}\Big),$$

which is spectrally equivalent to $\mathbf{K}$ uniformly in L and in k. We choose $\theta = \frac{1}{2}$ in the θ-scheme and the levels $L = 1, \ldots, 5$, yielding linear systems of size $N := \big(2^{L+1} - 1\big)\big(2^L - 1\big)$. In the θ-scheme, the nonsymmetric linear systems are solved by GMRES. The residuals r_ℓ of the ℓ-th GMRES step satisfy

$$\|r_\ell\| \le \big(1 - \alpha(\mathbf{K})^{-2}\big)^{\ell/2} \|r_0\|,$$

where the convergence measure is given by

$$\alpha(\mathbf{K}) = \frac{\|\mathbf{K}\|_2}{\lambda_{min}(\frac{1}{2}(\mathbf{K} + \mathbf{K}^\top))}.$$

Table 3 shows the values α for $\mathbf{K}$ with preconditioners of type I and II. Also in the nonselfadjoint case the preconditioner of type II appears to be best. We found this to hold over a wide range of parameters k, θ, r and m.

Table 3. GMRES convergence measure $\alpha(\mathbf{K})$ for preconditioners of type I and II

Level L	2	3	4	5
$\alpha\big(\mathbf{K}\big)$	2.890	8.264	43.76	200.0
$\alpha\big((\mathbf{C}^{\mathrm{I}})^{-1}\mathbf{K}(\mathbf{C}^{\mathrm{I}})^{-1}\big)$	2.82	8.264	25.0	47.16
$\alpha\big((\mathbf{C}^{\mathrm{II}})^{-1}\mathbf{K}(\mathbf{C}^{\mathrm{II}})^{-1}\big)$	2.13	3.77	6.75	11.62

A Appendix: Proofs of Lemmas 3.3, 3.4, 3.6 and 3.7

Proof of Lemma 3.3 Denote by $\Omega = \mathrm{supp}\psi^{l'}_{k'}$.
We write $u(x) = w^2(x)\psi^l_k(x)$ at $y = 2^{-l'}k'$ in the form

$$u(x) = u(y) + R^1 u(x), \quad R^1 u(x) = \int_y^x u'(\xi)\, \mathrm{d}\xi.$$

The remainder $R^1 u$ satisfies for $u \in W^{1,\infty}(\Omega)$ the estimate, cf. [4],

$$\| R^1 u \|_{L^\infty(\Omega)} \leq C \operatorname{diam}(\Omega) \mid u \mid_{W^{1,\infty}(\Omega)} .$$

Thus, there holds

$$\left| \frac{\int_0^1 w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x}{w(2^{-l}k)w(2^{-l'}k')} \right| = \left| \frac{\int_0^1 (u(y) + R^1 u(x))\psi_{k'}^{l'}(x)\,\mathrm{d}x}{w(2^{-l}k)w(2^{-l'}k')} \right|.$$

According to the vanishing moment condition, we can conclude

$$\begin{aligned}
\left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| &= \left| \frac{1}{w(2^{-l}k)w(2^{-l'}k')} \int_0^1 R^1 u(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x \right| \\
&\leq \frac{\| R^1 u \|_{L^\infty(\Omega)}}{w(2^{-l}k)w(2^{-l'}k')} \int_0^1 \left| \psi_{k'}^{l'}(x) \right| \,\mathrm{d}x \\
&\leq \operatorname{diam}(\Omega) \frac{\mid u \mid_{W^{1,\infty}(\Omega)}}{w(2^{-l}k)w(2^{-l'}k')} \int_0^1 \left| \psi_{k'}^{l'}(x) \right| \mathrm{d}x \\
&\leq C_\psi 2^{-l'} \frac{\mid u \mid_{W^{1,\infty}(\Omega)}}{w(2^{-l}k)w(2^{-l'}k')} 2^{-l'/2} .
\end{aligned}$$

Moreover, by $u(x) = w^2(x)\psi_k^l(x)$

$$\begin{aligned}
\frac{\mid u \mid_{W^{1,\infty}(\operatorname{supp}\psi_{k'}^{l'})}}{w(2^{-l}k)w(2^{-l'}k')} &= \frac{C_\psi}{w(2^{-l}k)w(2^{-l'}k')} \| (w^2)'\psi_k^l + w^2(\psi_k^l)' \|_{L^\infty(\operatorname{supp}\psi_{k'}^{l'})} \\
&\leq \frac{C_\psi}{w(2^{-l}k)w(2^{-l'}k')} \left\{ \| (w^2)' \|_{L^\infty} 2^{\frac{l}{2}} + \| w^2 \|_{L^\infty} 2^{\frac{3l}{2}} \right\}.
\end{aligned}$$

Due to Lemma 3.2 and Lemma 3.1, we estimate

$$\frac{\| (w^2)' \|_{L^\infty}}{w(2^{-l}k)w(2^{-l'}k')} \leq 2^l C_w \quad \text{and} \quad \frac{\| w^2 \|_{L^\infty}}{w(2^{-l}k)w(2^{-l'}k')} \leq C_w,$$

which gives the desired result. □

Proof of Lemma 3.4 We develop $u(x) = w^2(x)\psi_k^l(x)$ around $y = 2^{-l'}k'$ in a Taylor series:

$$u(x) = w^2(x)\psi_k^l(x) = w^2(y)\psi_k^l(y) + R_1 u(x).$$

According to the vanishing moment $\int_0^1 \psi_{k'}^{l'}(x)\,\mathrm{d}x = 0$ we obtain

$$\int_0^1 w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x = \int_{\operatorname{supp}\psi_{k'}^{l'}} R_1 u(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x. \tag{A.1}$$

We note that, for $x \in [0, 2^{-l}]$

$$|\psi_k^l(x)| \le C_\psi 2^{\frac{l}{2}} (2^l x)^\beta,$$

cf. (3.2) and

$$|(\psi_k^l)'(x)| \le C_\psi 2^{\frac{l}{2}(1+2\beta)} x^{\beta-1}. \tag{A.2}$$

Inserting this fact and $|(w^2)'(x)| \le C_w x^{2\alpha-1}$ into the relation (A.1) we get

$$\begin{aligned} I := \int_0^1 w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)\,\mathrm{d}x &\le \| R_1 u \|_{L^\infty(\operatorname{supp}\psi_{k'}^{l'})} \int_0^1 |\psi_{k'}^{l'}(x)|\,\mathrm{d}x \\ &\le C_\psi 2^{-3l'/2} \| (w^2)'\psi_k^l + (\psi_k^l)' w^2 \|_{L^\infty} \\ &\le C_\psi C_w \left| (2^{-l'}k')^{2\alpha+\beta-1} 2^{-\frac{3}{2}l'} 2^{\frac{l}{2}(1+2\beta)} \right| \end{aligned}$$

due to the assumption $0 \notin \operatorname{supp}\psi_{k'}^{l'}$. Since $0 \in \operatorname{supp}\psi_k^l$, there holds $k \approx 1$ or, equivalently, $2^{-l}k \approx 2^{-l}$. Inserting the above results, we obtain

$$\begin{aligned} \left| \left(\psi_k^l, \psi_{k'}^{l'}\right)_w \right| &= \frac{I}{w(2^{-l}k)w(2^{-l'}k')} = \frac{I}{(2^{-l}k)^\alpha (2^{-l'}k')^\alpha} \\ &\le C_w \frac{I}{2^{-l\alpha}(2^{-l'}k')^\alpha} \\ &\le C_w C_\psi \left| (2^{-l'}k')^{\alpha+\beta-1} 2^{-\frac{3}{2}l'} 2^{\frac{l}{2}(1+2\beta+2\alpha)} \right|. \end{aligned}$$

Finally, we obtain

$$\left| \left(\psi_k^l, \psi_{k'}^{l'}\right)_w \right| \le C_\psi C_w \left| 2^{-\frac{1}{2}|l'-l|(1+2\alpha+2\beta)} k'^{\alpha+\beta-1} \right|,$$

which is the desired result. □

Proof of Lemma 3.6 We split

$$\begin{aligned} &\left| \left(\psi_k^l, \psi_{k'}^{l'}\right)_w \right| \\ &= \left| \int_0^{2^{-l'}} \frac{w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x + \int_{2^{-l'}}^{2^{-l'}N} \frac{w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right| \\ &\le \left| \int_0^{2^{-l'}} \frac{w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right| + \left| \int_{2^{-l'}}^{2^{-l'}N} \frac{w^2(x)\psi_k^l(x)\psi_{k'}^{l'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right|. \end{aligned} \tag{A.3}$$

We estimate now the first integral on the right hand side of (A.3). From Assumption 3.2 and $0 \in \operatorname{supp}\psi_k^l \cap \operatorname{supp}\psi_{k'}^{l'}$ we have

$$|\psi_k^l(x)| \le c 2^{\frac{l}{2}} (2^l x)^\beta \le c 2^{\frac{l}{2}(1+2\beta)} x^\beta \quad \text{for} \quad x \in [0, 2^{-l}]$$

and $|\psi^{l'}_{k'}(x)| \le 2^{\frac{l'}{2}(1+2\beta)} x^{\beta}$ for $x \in [0, 2^{-l'}]$. Therefore, using the estimate $w^2(x) \le c x^{2\alpha}$ we deduce the bound

$$\begin{aligned}\left| \int_0^{2^{-l'}} w^2(x)\psi^l_k(x)\psi^{l'}_{k'}(x)\,\mathrm{d}x \right| &\le c 2^{\frac{l+l'}{2}(1+2\beta)} \int_0^{2^{-l'}} x^{2\alpha+2\beta}\,\mathrm{d}x \\ &= c 2^{\frac{l+l'}{2}(1+2\beta)} 2^{-l'(1+2\beta+2\alpha)}\end{aligned}$$

if $2\alpha + 2\beta > -1$, cf. Assumption 3.2. Otherwise this integral does not exist. Furthermore, from $0 \in \operatorname{supp} \psi^l_k$ and $0 \in \operatorname{supp} \psi^{l'}_{k'}$, we can conclude $2^{-l}k \sim 2^{-l}$ and $2^{-l'}k' \sim 2^{-l'}$. Hence,

$$\begin{aligned}\left| \int_0^{2^{-l'}} \frac{w^2(x)\psi^l_k(x)\psi^{l'}_{k'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right| &\le c 2^{\frac{l+l'}{2}(1+2\beta+2\alpha)} 2^{-l'(1+2\beta+2\alpha)} \\ &= c 2^{\frac{l-l'}{2}(1+2\beta+2\alpha)}. \qquad \text{(A.4)}\end{aligned}$$

We estimate now the second sum on the right hand side of (A.3). By $w(x) \asymp w(2^{-l'}k') \asymp w(2^{-l'})$ for all $x \in \operatorname{supp} \psi^{l'}_{k'} \backslash [0, 2^{-l'})$ and $w(2^{-l}k) \asymp w(2^{-l})$ we have

$$\begin{aligned}\left| \int_{2^{-l'}}^{2^{-l'}N} \frac{w^2(x)\psi^l_k(x)\psi^{l'}_{k'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right| &\le c \left| \int_{2^{-l'}}^{2^{-l'}N} \frac{w(x)}{2^{-l\alpha}} \psi^l_k(x)\psi^{l'}_{k'}(x)\,\mathrm{d}x \right| \\ &\le c 2^{l\alpha} 2^{\frac{l'}{2}} \left| \int_{2^{-l'}}^{2^{-l'}N} w(x)\psi^l_k(x)\,\mathrm{d}x \right|.\end{aligned}$$

Now apply $w(x) \le c x^{\alpha}$ and $|\psi^l_k(x)| \le c 2^{\frac{l}{2}(1+2\beta)} x^{\beta}$. The integrals yield the following estimate

$$\left| \int_{2^{-l'}}^{2^{-l'}N} \frac{w^2(x)\psi^l_k(x)\psi^{l'}_{k'}(x)}{w(2^{-l}k)w(2^{-l'}k')}\,\mathrm{d}x \right| \le c 2^{\frac{l-l'}{2}(1+2\alpha+2\beta)}. \qquad \text{(A.5)}$$

Inserting (A.4) and (A.5) into (A.3) proves the lemma. □

Proof of Lemma 3.7 We note that for $l > l'$ holds

$$\left| \left(\psi^l_k, \psi^{l'}_{k'}\right)_w \right| = 0 \quad k \in \nabla^L_l, k' \in \nabla^I_{l'}, \qquad \text{(A.6)}$$

cf. (3.4). Then, we can conclude

$$\sum_{l=1}^{\infty} \sum_{k \in \nabla^L_l} \left| \left(\psi^l_k, \psi^{l'}_{k'}\right)_w \right| = \sum_{l=1}^{l'} \sum_{k \in \nabla^L_l} \left| \left(\psi^l_k, \psi^{l'}_{k'}\right)_w \right|.$$

Using Proposition 3.2, we note the second summation $\sum_{k\in\nabla_l^L}$ has only $\mathcal{O}(1)$ nonzero summands. We distinguish now the two cases $1 < k' < 2^{l'-l}$ and $k' \geq 2^{l'-l}$. We start with $1 < k' < 2^{l'-l}$ and obtain by Lemma 3.4

$$\sum_{l=1}^{l'} \sum_{k} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \sum_{l=1}^{l'} 2^{-\frac{1}{2}(l'-l)(1+2\alpha+2\beta)} (k')^{\alpha+\beta-1}.$$

If $\alpha + \beta \geq 1$ then $(k')^{\alpha+\beta-1} \leq (2^{l'-l})^{\alpha+\beta-1}$. Then, we can conclude

$$\sum_{l=1}^{l'} \sum_{k} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \sum_{l=1}^{l'} 2^{\frac{3}{2}(l-l')} \leq c. \tag{A.7}$$

In the case $\alpha + \beta < 1$ we estimate $(k')^{\alpha+\beta-1} \leq 1$ and obtain by the geometric series

$$\sum_{l=1}^{l'} \sum_{k} \left| \left(\psi_k^l, \psi_{k'}^{l'} \right)_w \right| \leq c \tag{A.8}$$

if $2\alpha + 2\beta + 1 > 0$. If $k' \geq 2^{l'-l}$ we obtain using Lemma 3.3 the estimate (A.7) directly for all $\alpha, \beta \in \mathbb{R}$. □

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